A Pragmatic Interpretation of Quantum Logic

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Abstract

Scholars have wondered for a long time whether quantum mechanics (QM) subtends a quantum concept of truth which originates quantum logic (QL) and is radically different from the classical (Tarskian) concept of truth. We show in this paper that QL can be interpreted as a pragmatic language $\mathcal{L}_{QD}^{P}$ of pragmatically decidable assertive formulas, which formalize statements about physical systems that are empirically *justified* or *unjustified* in the framework of QM. According to this interpretation, QL formalizes properties of the metalinguistic concept of empirical justification within QM rather than properties of a quantum concept of truth. This conclusion agrees with a general integrationist perspective, according to which nonstandard logics can be interpreted as theories of metalinguistic concepts different from truth, avoiding competition with classical notions and preserving the globality of logic. By the way, some elucidations of the standard concept of quantum truth are also obtained.

Key words: pragmatics, quantum logic, quantum mechanics, justifiability, decidability, global pluralism.

1 Introduction

The formal structure called *quantum logic* (QL) springs out in a natural way from the formalism of quantum mechanics (QM). Scholars have debated for a long time on it, wondering whether it subtends a concept of quantum truth which is typical of QM, and a huge literature exists on this topic. We limit ourselves here to quote the classical book by Jammer,(1) which provides a general review of QL from its birth to the early seventies, and the recent books by Rèdei(2) and Dalla Chiara *et al.*,(3) which contain updated bibliographies.

Whenever the existence of a quantum concept of truth is accepted, one sees at once that it has to be radically different from the classical (Tarskian) concept, since the set of propositions of QL has an algebraic structure which is different from the structure of classical propositional logic. Thus, a new problem arises, *i.e.* the problem of the "correct" logic to be adopted when reasoning in QM.

We want to show in the present paper that the above problem can be avoided by adopting an *integrated perspective*, which preserves both the globality of logic (in the sense of *global pluralism*, which admits the existence of a plurality of mutually compatible logical systems, but not of systems which are mutually incompatible[4]) and the classical notion of *truth as correspondence*, which we consider as explicated rigorously by Tarski's semantic theory.[5,6] This perspective reconciliates non-Tarskian theories of truth with Tarski's theory by reinterpreting them as theories of metalinguistic concepts that are different from truth, and can be fruitfully applied to QL. Indeed, we prove in this paper that QL can be interpreted as a theory of the concept of *empirical justification* within QM.

In order to grasp intuitively our results, let us anticipate briefly some remarks that will be discussed more extensively in Sec. 2.

First of all, it must be noted that QM usually avoids making statements about properties of individual samples of a physical system (*physical objects*). Rather, it is concerned with probabilities of results of measurements on physical objects (*standard interpretation*, as espounded in any manual of QM; see, *e.g.*, Refs. 7, 8 and 9), or with statistical predictions about ensembles of identically prepared physical objects (*statistical interpretation*; see, *e.g.*, Refs. 1, 10 and 11). Yet, QM also distinguishes between properties that are real (or *actual*) and properties that are not real (or *potential*) in a given state S of the physical system that is considered (briefly, the property E is actual in S whenever a test of E on any physical object x in S would show that E is possessed by x without changing S[12]). This amounts to introduce implicitly a concept of truth that also applies to statements about individuals. Indeed, asserting that a property E is actual in the state S is equivalent to asserting that the statement $E(x)$ that attributes E to a physical object x is *true* for every x in the state S. Moreover, according to QM, $E(x)$ is true, for a given x in the state S, if and only if (briefly, *iff*) E is actual in the state S.[12] Falsity is then defined by considering a complementary property $E^{\perp}$ of E, so that $E(x)$ is false for a given x in the state S iff $E^{\perp}$ is actual in S. It follows in particular that $E(x)$ is true (false) for a given x in the state S iff it is true (false) for every x in S, or, equivalently, iff it is *certainly true* (*certainly false*) in S. This result explains the notion of true as *certain* introduced in some well known approaches to QM[13,14]. More important, it shows that the notion of truth has very peculiar features in QM. Indeed, the truth and falsity of a statement $E(x)$ about an individual are equivalent to the truth of two universally quantified statements. Both these statements may be false. In this case $E(x)$ has no truth value, hence it is meaningless. The existence of meaningless statements implies, in particular, that no Tarskian set-theoretical semantics can be introduced in QM.

The quantum notion of truth and meaning pointed out above is typical of the standard interpretation of QM, and it is inspired by a verificationist position which identifies truth and verifiability, meaning and verifiability conditions. These identifications are rather doubtful from an epistemological viewpoint, yet it is commonly maintained in the literature that the standard quantum conception of truth has no alternatives, since it seems firmly rooted in the formalism of QM itself. The mathematical apparatus of QM would imply indeed the impos-

sibility of defining an assignment function associating a truth value with every individual statement of the form $E(x)$ by referring only to the property E and the state S of x. The outcomes obtained in a concrete experiment whenever E or $E^{\perp}$ are not actual in S would depend on the set of observations that are carried out simultaneously, not only on S (*contextuality*).[15−18]

Notwithstanding the arguments supporting it, the standard viewpoint can be criticized, and an alternative *SR interpretation* of QM can be constructed based on an epistemological position (*semantic realism*, or, briefly *SR*) which allows one to define a truth value for every statement of the form $E(x)$ according to a Tarskian set-theoretical model.[19−26] Of course, all statements that are certainly true (equivalently, true) or certainly false (equivalently, false) according to the standard interpretation with its quantum concept of truth, are also certainly true or certainly false, respectively, according to the SR interpretation with its Tarskian concept of truth. The remaining statements are meaningless according to the former interpretation, while they have truth values according to the latter: these values, however, may change when different objects in the same state are considered, and cannot be predicted in QM (which is, in this sense, an incomplete theory).

Because of its intuitive, philosophical and technical advantages, we adopt the SR interpretation in the present paper. It is then important to observe that our definitions and reasonings take into account only statements that are certainly true (certainly false) in the sense explained above, hence they actually do not depend on the choice of the interpretation of QM (standard or SR). Thus, our reinterpretation of QL should be acceptable also for logicians and physicists who do not agree with our epistemological position. Of course, if the SR interpretation is not accepted one loses all philosophical advantages of the integrated perspective mentioned at the beginning of this section.

Let us come now to empirical justification. Whenever a statement $E(x)$ is certainly true (certainly false), its truth (falsity) can be predicted within QM if the property E and the state S of x are known, and can be checked (by means of nontrivial physical procedures, see Sec. 2.6). Hence, we can say that the assertion of $E(x)$ ($E^{\perp}(x)$) is empirically justified, since we can both deduce the truth of $E(x)$ ($E^{\perp}(x)$) inside QM and provide an empirical proof of it. More formally, one can introduce an assertion sign $\vdash$ and say that $E(x)$ is certainly true (certainly false) iff $\vdash E(x)$ ($\vdash E^{\perp}(x)$) is empirically justified. In this way a semantic notion (certainty of truth) is translated into a pragmatic notion (empirical justification). Now, we remind that a pragmatic extension of a classical language and some general properties of the concept of justification have been studied by Dalla Pozza and by the author[27] and note that all results obtained in this research apply to the notion of empirical justification introduced above. Moreover, further results can be obtained which are typical of the case under consideration, since the notion of justification is now specified (empirical justification in QM). Thus, a pragmatic language $\mathcal{L}_Q^P$ can be constructed (Sec. 3) in which assertions of the form $\vdash E(x)$ are taken as elementary *assertive formulas* (*afs*) and pragmatic connectives are introduced, for which a *set-theoretical pragmatics* is defined basing on the concept of empirical justification in QM.

This pragmatics defines a justification value for every elementary or complex af of $\mathcal{L}_Q^P$, yet not all complex afs of $\mathcal{L}_Q^P$ are *pragmatically decidable*, that is, such that an empirical procedure of justification exists (it obviously exists for all elementary afs of $\mathcal{L}_Q^P$ because of our arguments above). However, one can single out a subset of pragmatically decidable afs of $\mathcal{L}_Q^P$ and consider a sublanguage $\mathcal{L}_{QD}^P$ of $\mathcal{L}_Q^P$ which contains only afs in this subset. It is then easy to see that our set-theoretical pragmatics, when restricted to $\mathcal{L}_{QD}^P$, endows it with the structure of QL.

The above result is highly interesting in our opinion. Indeed, it provides the desired reinterpretation of QL as a theory of the metalinguistic concept of empirical justification in QM, allowing us to place it within an integrationist perspective that avoids any conflict with classical logic (we stress again that this conclusion can be accepted also by scholars who want to maintain the standard interpretation of QM).

We conclude this Introduction by observing that our results suggest that the standard partition of properties in two subsets (actual properties and potential properties) should be substituted by a partition in three subsets, as follows.

Actual properties. A property E is actual in the state S iff the assertion $\vdash E(x)$, with x in S, is justified.

Nonactual properties. A property E is nonactual in the state S iff the assertion $\vdash E^{\perp}(x)$, with x in S, is justified.

Potential properties. A property E is potential in the state S iff both assertions $\vdash E(x)$ and $\vdash E^{\perp}(x)$, with x in S, are unjustified.

2 Physical preliminaries

We introduce in this section a number of symbols, definitions and physical concepts that will be extensively used in Sec. 3 in order to supply an intuitive support and an intended interpretation for the pragmatic language that will be introduced there.

2.1 Basic notions and mathematical representations

The following notions will be taken as primitive.

Physical system Ω.

Pure state S *of* Ω, and *set* $\mathcal{S}$ *of all pure states of* Ω (the word *pure* will be usually implied in the following).

Testable property E *of* Ω, and *set* $\mathcal{E}$ *of all testable properties of* Ω (the word *testable* will be usually implied in the following).[1]

States and properties will be interpreted operationally as follows.

[1]It must be noted that the physical properties considered here are first order properties from a logical viewpoint.[(26)] Higher order properties obviously occur in physics and will be encountered later on (Sec. 2.6), but they need not be considered here.

A state $S \in \mathcal{S}$ is a class of physically equivalent[2] preparing devices (briefly, *preparations*) which may prepare individual samples of Ω (*physical objects*). A physical object x *is in the state* S iff it is prepared by a preparation $\pi \in S$.

A property $E \in \mathcal{E}$ is a class of physically equivalent ideal dichotomic (outcomes 1, 0) registering devices (briefly, *registrations*) which may test physical objects.[3]

The above notions do not distinguish between classical and quantum mechanics. The mathematical representation of physical systems, states and properties are different, however, in the two theories. Let us resume these representations in the case of QM.

Every physical system Ω is associated with a Hilbert space $\mathcal{H}$ over the field of complex numbers (we use the Dirac notation $| \cdot \rangle$ in order to denote vectors of $\mathcal{H}$ in the following).

Let us denote by $(\mathcal{L}(\mathcal{H}), \subset)$ the partially ordered set (briefly, *poset*) of all closed subspaces of $\mathcal{H}$ (here $\subset$ denotes set-theoretical inclusion), and let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be the set of all one-dimensional subspaces of $\mathcal{H}$. Then (in absence of superselection rules) a mapping

$\varphi : S \in \mathcal{S} \longrightarrow \varphi(S) \in \mathcal{A}$

exists which maps bijectively the set $\mathcal{S}$ of all states of Ω onto $\mathcal{A}$,[4] and a mapping

$\chi : E \in \mathcal{E} \longrightarrow \chi(E) \in \mathcal{L}(\mathcal{H})$

exists which maps bijectively the set $\mathcal{E}$ of all properties of Ω onto $\mathcal{L}(\mathcal{H})$.[5]

2.2 Physical Quantum Logic

The poset $(\mathcal{L}(\mathcal{H}), \subset)$ is characterized by a set of mathematical properties. In particular, it is a complete, orthocomplemented, weakly modular, atomic lattice which satisfies the covering law[(13,27–30)]. We denote by $^\perp$, $\Cap$ and $\Cup$ orthocomplementation, meet and join, respectively, in $(\mathcal{L}(\mathcal{H}), \subset)$, and remind that $\Cap$ coincides with the set-theoretical intersection $\cap$ of subspaces of $\mathcal{H}$, while $^\perp$ does not generally coincide with the set-theoretical complementation $'$, nor $\Cup$ coincides with the set-theoretical union $\cup$. Furthermore, we denote the minimal element $\{| 0 \rangle\}$ and the maximal element $\mathcal{H}$ of $(\mathcal{L}(\mathcal{H}), \subset)$ by O and I, respectively. Finally, we note that $\mathcal{A}$ obviously coincides with the set of all atoms of $(\mathcal{L}(\mathcal{H}), \subset)$.

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[2]The notion of physical equivalence for preparing or registering devices is not trivial.[(11,21)] We do not discuss it here for the sake of brevity.

[3]Note that a registration may act as a new preparation of the physical object x, so that the state of x may change after a test on it.

[4]It follows easily that every state S can also be represented by any vector $| \psi \rangle \in \varphi(S) \in \mathcal{A}$, which is the standard representation adopted in elementary QM. Moreover, a state S is usually represented by an (orthogonal) projection operator on $\varphi(S)$ in more advanced QM. However, the representation φ introduced here is more suitable for our purposes in the present paper.

[5]Equivalently, a property is often represented in QM as a pair (A, Δ), where is A a self-adjoint operator on $\mathcal{H}$ representing a physical observable, and Δ a Borel set on the real line.[(28)] We do not use this representation, however, in the present paper.

Let us denote by $\prec$ the order induced on $\mathcal{E}$, via the bijective representation χ, by the order $\subset$ defined on $\mathcal{L}(\mathcal{H})$. Then, the poset $(\mathcal{E}, \prec)$ is order-isomorphic to $(\mathcal{L}(\mathcal{H}), \subset)$, hence it is characterized by the same mathematical properties characterizing $(\mathcal{L}(\mathcal{H}), \subset)$. In particular, the unary operation induced on it, via χ, by the orthocomplementation defined on $(\mathcal{L}(\mathcal{H}), \subset)$, is an orthocomplementation, and $(\mathcal{E}, \prec)$ is an orthomodular (i.e., orthocomplemented and weakly modular) lattice, usually called *the lattice of properties* of Ω. By abuse of language, we denote the lattice operations on $(\mathcal{E}, \prec)$ by the same symbols used above in order to denote the corresponding lattice operations on $(\mathcal{L}(\mathcal{H}), \subset)$.

Orthomodular lattices are said to characterize semantically *orthomodular QLs* in the literature.(3) The lattice of properties has a less general structure in QM, since it inherits a number of further properties from $(\mathcal{L}(\mathcal{H}), \subset)$. Therefore, $(\mathcal{E}, \prec)$ will be called *physical QL* in this paper.

A further lattice, isomorphic to $(\mathcal{E}, \prec)$, will be used in the following. In order to introduce it, let us consider the mapping

$\rho : E \in \mathcal{E} \longrightarrow \mathcal{S}_E = \{S \in \mathcal{S} \mid \varphi(S) \subset \chi(E)\} \in \mathcal{L}(\mathcal{S})$,

where $\mathcal{L}(\mathcal{S}) = \{\mathcal{S}_E \mid E \in \mathcal{E}\}$ is the range of ρ, which generally is a proper subset of the power set $\mathcal{P}(\mathcal{S})$ of $\mathcal{S}$. The poset $(\mathcal{L}(\mathcal{S}), \subset)$ is order-isomorphic to $(\mathcal{L}(\mathcal{H}), \subset)$, hence to $(\mathcal{E}, \prec)$, since φ and χ are bijective, so that ρ is bijective and order-preserving. Therefore $(\mathcal{L}(\mathcal{S}), \subset)$ is characterized by the same mathematical properties characterizing $(\mathcal{E}, \prec)$. In particular, the unary operation induced on it, via ρ, by the orthocomplementation defined on $(\mathcal{E}, \prec)$, is an orthocomplementation, and $(\mathcal{L}(\mathcal{S}), \subset)$ is an orthomodular lattice. We denote orthocomplementation, meet and join on $(\mathcal{L}(\mathcal{S}), \subset)$ by the same symbols $^\perp$, $\Cap$, and $\Cup$, respectively, that we have used in order to denote the corresponding operations on $(\mathcal{L}(\mathcal{H}), \subset)$ and $(\mathcal{E}, \prec)$, and call $(\mathcal{L}(\mathcal{S}), \subset)$ *the lattice of closed subsets of* $\mathcal{S}$ (the word *closed* refers here to the fact that, for every $\mathcal{S}_E \in \mathcal{L}(\mathcal{S})$, $(\mathcal{S}_E^\perp)^\perp = \mathcal{S}_E$). We also note that the operation $\Cap$ coincides with the set-theoretical intersection $\cap$ on $\mathcal{L}(\mathcal{S})$ because of the analogous result holding in $(\mathcal{L}(\mathcal{H}), \subset)$.[6]

To close up, let us observe that the unary operation $^\perp$ defined on $\mathcal{L}(\mathcal{S})$ can be extended to $\mathcal{P}(\mathcal{S})$ by setting, for every $\mathcal{T} \in \mathcal{P}(\mathcal{S})$,

$\mathcal{T}^\perp = (min\{\mathcal{S}_E \in \mathcal{L}(\mathcal{S}) \mid \mathcal{T} \subset \mathcal{S}_E\})^\perp$

(the symbol *min* obviously refers to the order $\subset$ defined on $\mathcal{L}(\mathcal{S})$). This extension will be needed indeed in Sec. 3.2.

2.3 Actual and potential properties

We say that a property E is *actual* (*nonactual*) in the state S iff one can perform a test of E on any physical object x in the state S by means of a registration

[6] Whenever the dimension of $\mathcal{H}$ is finite, the lattice $(\mathcal{L}(\mathcal{H}), \subset)$ and/or the lattice $(\mathcal{L}(\mathcal{S}), \subset)$ can be identified with Birkhoff and von Neumann's lattice of *experimental propositions*, which was introduced in the 1936 paper that started the research on QL.(31) This identification is impossible, however, if $\mathcal{H}$ is not finite-dimensional, since Birkhoff and von Neumann's lattice is modular, not simply weakly modular. The requirement of modularity has deep roots in the von Neumann concept of probability in QM according to some authors.(2)

$r \in E$, obtaining outcome 1 (0) without modifying S.[7]

Basing on the above definition, for every state $S \in \mathcal{S}$ three subsets of $\mathcal{E}$ can be introduced.

$\mathcal{E}_S$: the set of all properties that are actual in S.

$\mathcal{E}_S^{\perp}$: the set of all properties that are nonactual in S.

$\mathcal{E}_S^I$: the set $\mathcal{E} \setminus \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$ (called the set of all properties that are *indeterminate*, or *potential*, in S).

By using the mathematical apparatus of QM, the sets $\mathcal{E}_S$ and $\mathcal{E}_S^{\perp}$ can be characterized as follows.

$\mathcal{E}_S = \{E \in \mathcal{E} \mid \varphi(S) \subset \chi(E)\} = \{E \in \mathcal{E} \mid S \in \mathcal{S}_E\}$.

$\mathcal{E}_S^{\perp} = \{E \in \mathcal{E} \mid \varphi(S) \subset \chi(E)^{\perp}\} = \{E \in \mathcal{E} \mid S \in \mathcal{S}_E^{\perp}\}$.

It can also be proved that $\mathcal{E}_S$ ($\mathcal{E}_S^{\perp}$) coincides with the set of all properties that have probability 1 (0), according to QM, for every x in the state S, and that the set $\mathcal{E}_S^I$ (which is non-void in QM, while it would be void in classical physics) coincides with the set of all properties that have probability different from 0 and 1 for every x in the state S.

Further characterizations of the above sets can be obtained as follows.[(12)]

Since the mapping ρ is bijective, while every singleton $\{S\}$, with $S \in \mathcal{S}$, obviously is an atom of $\mathcal{L}(\mathcal{S})$, one can associate a property $E_S = \rho^{-1}(\{S\})$ (equivalently, $E_S = \chi^{-1}(\varphi(S))$) with every $S \in \mathcal{S}$. This property is an atom of $(\mathcal{E}, \prec)$, and is usually called the *support* of S. The mapping ρ^{-1} thus induces a one-to-one correspondence between (pure) states and atoms of $(\mathcal{E}, \prec)$. Then, one can prove the following equalities.

$\mathcal{E}_S = \{E \in \mathcal{E} \mid E_S \prec E\}$.

$\mathcal{E}_S^{\perp} = \{E \in \mathcal{E} \mid E \prec E_S^{\perp}\}$.

$\mathcal{E}_S^I = \{E \in \mathcal{E} \mid E_S \not\prec E$ and $E \not\prec E_S^{\perp}\}$.

Finally, the following equality also follows from the above definitions.

$\mathcal{S}_E = \{S \in \mathcal{S} \mid E_S \prec E\}$.

2.4 Truth in standard QM

No mention has been done of truth values (*true/false*) in the foregoing sections. However, we will be concerned with logical structures in Sec. 3, hence it is natural to wonder what QM says about the truth of a sentence as "the physical object x has the property E" (briefly, $E(x)$ in the following).

We have already noted in the Introduction that QM usually avoids making explicit statements regarding individual samples of physical systems. Yet, a sentence as "the property E is actual in the state S" (Sec. 2.3) intuitively

[7] One can provide an intuitive support to this definition by noticing that the result obtained in a test of E on a physical object x in the state S can be attributed to x only whenever S is not modified by the test. Moreover, only in this case the test is *repeatable*, i.e., it can be performed again obtaining the same result.

It is well known that classical physics assumes that tests which do not modify the state S are always possible, at least as ideal limits of concrete procedures, while this assumption does not hold in QM.

means that all physical objects in the state S have the property E. Hence, it can be translated, in terms of truth, into the sentence "for every physical object x in the state S, $E(x)$ is true". This translation shows that QM is concerned also with truth values of individual statements. Moreover, by considering the literature on the subject, one can argue that QM more or less implicitly adopts the following verificationist criterion of truth.[12]

EV (empirical verificationism). *The sentence $E(x)$ has truth value* true *(*false*) for a physical object x in the state S iff E is actual (nonactual) in S, while it is meaningless otherwise.*

Criterion EV is obviously at odds with standard definitions in classical logic (CL), and is suggested by the fact that E can be attributed (not attributed) to a physical object x in the state S on the basis of an experimental procedure only when it is actual (nonactual) for x (see Sec. 2.6). Hence, we say that $E(x)$ is *Q-true* (*Q-false*) whenever its truth value is true (false) according to criterion EV, in order to stress the difference between the truth values introduced in QM and those introduced in CL.

Because of the foregoing translation, criterion EV implies the following proposition.

TF. *The sentence $E(x)$ is* Q-true *(*Q-false*) for a physical object x in the state S iff it is* Q-true *(*Q-false*) for every physical object x in the state S.*

Loosely speaking, proposition TF can be rephrased by saying that $E(x)$ is true (false) in the sense established by criterion EV iff it is *certainly* true (*certainly* false) in the same sense, which explains the intuitive terminology that we have adopted in the Introduction.

Furthermore, criterion EV implies that $E(x)$ has a truth value in standard QM iff $E \in \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$ (of course, $E(x)$ is Q-true iff $E \in \mathcal{E}_S$, Q-false iff $E \in \mathcal{E}_S^{\perp}$). It is then important to observe that the characterizations of $\mathcal{E}_S$ and $\mathcal{E}_S^{\perp}$ provided in Sec. 2.3 show that, for every $S \in \mathcal{S}$, one can deduce from theoretical laws of QM whether a property E belongs to $\mathcal{E}_S \cup \mathcal{E}_S^{\perp}$. In particular, E belongs to $\mathcal{E}_S$ ($\mathcal{E}_S^{\perp}$) iff it has probability 1 (0) for every x in the state S. Hence, one can predict, for every $E \in \mathcal{E}$ and x in the state S, whether $E(x)$ is Q-true, Q-false or meaningless. This result shows that standard QM is a semantically complete theory[12] and, together with proposition TF, explains the definition of *true* as *certain*, or *predictable*, which occurs in some approaches to QM.[13,14]

2.5 Nonobjectivity versus objectivity in QM

The position expounded in Sec. 2.4 about the truth value of sentences of the form $E(x)$, with $E \in \mathcal{E}$, is sometimes summarized by saying, briefly, that physical properties are *nonobjective* in standard QM (to be precise, only the properties in $\mathcal{E}_S^I$ should be classified as nonobjective for a given $S \in \mathcal{S}$).

Nonobjectivity of properties is supported by a number of arguments. Some of them are based on empirical results (e.g., the two-slits experiment), some follow from seemingly reasonable epistemological choices (e.g., the adoption of a verificationist position, together with the indeterminacy principle) and some

take the form of theorems deduced from the mathematical apparatus of QM. These last arguments are usually considered conclusive in the literature. We remind here the Bell-Kochen-Specker and Bell's theorems[15−18] which seem to prove that it is impossible to assign classical truth values to all sentences of the form $E(x)$, with $E \in \mathcal{E}$, without contradicting the predictions of QM.

However, all arguments which show that nonobjectivity of properties is an unavoidable feature of QM can be criticized (this of course does not make the claim of nonobjectivity wrong, but only proves that there are alternatives to it). In particular, one can observe that a *no-go theorem* as Bell-Kochen-Specker's is certainly correct from a mathematical viewpoint, but rests on implicit assumptions that are problematic from a physical and epistemological viewpoint.[22−25] Basing on this criticism, an alternative interpretation (*semantic realism*, or *SR*, interpretation) has been propounded by the author, together with other authors.[19−23,25,26] As we have already observed in the Introduction, the SR interpretation adopts a Tarskian theory of truth as correspondence, and all properties are objective according to it (equivalently, the sentence $E(x)$ has a truth value defined in a classical way for every physical object x and property E). According to this interpretation $E(x)$ is *certainly true* (*certainly false*) in the state S, that is, it is true (false) in a classical sense for every x is in the state S, iff $E \in \mathcal{E}_S$ ($E \in \mathcal{E}_S^{\perp}$), hence iff it is Q-true (Q-false) according to the standard interpretation.

The SR interpretation of QM has some definite advantages. Firstly, it makes QM compatible with a realistic perspective without requiring any change of its mathematical apparatus and preserving all statistical predictions following from the standard interpretation, hence it provides a solution of the quantum measurement problem.[26] Secondly, it rests on a classical conception of truth and meaning. Thirdly, it leads one to consider QM as an incomplete theory,[12] and provides some suggestions about the way in which a more general theory embodying QM could be constructed.

Also within the SR interpretation one can deduce from theoretical laws of QM whether $E \in \mathcal{E}_S$ ($E \in \mathcal{E}_S^{\perp}$), for a given $S \in \mathcal{S}$. Moreover, for every $E \in \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$, the sentence $E(x)$ obviously is certainly true, hence true (certainly false, hence false) iff $E \in \mathcal{E}_S$ ($E \in \mathcal{E}_S^{\perp}$). On the contrary, no prediction of the truth value of $E(x)$ can be done if $E \notin \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$. Thus, the difference between the standard and the SR interpretation reduces to the fact that, whenever $E \in \mathcal{E}_S^{I}$, $E(x)$ is meaningless within the former, while it has a truth value that cannot be predicted by QM within the latter.

2.6 Empirical proof in QM

The results at the end of Secs. 2.4 and 2.5 show that, whenever x is in the state S, the truth value of the sentence $E(x)$ can be predicted (or *theoretically proved*) iff $E \in \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$, both in the standard and in the SR interpretation. One is thus led to wonder whether and when the truth value of $E(x)$ can be determined empirically. At first glance, it seems sufficient to test x by means of a registering device belonging to E (Sec. 2.1). This is untrue according

to the standard as well as the SR interpretation. Indeed, both interpretations maintain that a single test modifies, whenever $E \notin \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$, the state S of the physical object x, so that its result refers to the final state after the test, which is different from S (moreover, within the standard interpretation, $E(x)$ has no truth value whenever $E \notin \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$). Thus, a test of $E(x)$ is physically meaningful iff $E \in \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$, since only in this case it does not modify the state S. It follows that an *empirical proof* of the truth value of $E(x)$ can be given iff a theoretical proof of this value exists, and it consists in checking whether $E \in \mathcal{E}_S$ or $E \in \mathcal{E}_S^{\perp}$. Then, the characterizations of $\mathcal{E}_S$ and $\mathcal{E}_S^{\perp}$ in Sec. 2.3 suggest the empirical procedures to be adopted. Indeed, they show that $E \in \mathcal{E}_S$ ($E \in \mathcal{E}_S^{\perp}$) iff $E_S \prec E$ ($E \prec E_S^{\perp}$), or, equivalently, iff $S \in \mathcal{S}_E$ ($S \in \mathcal{S}_E^{\perp}$). Hence, one can get an empirical proof that $E(x)$ is Q-true (Q-false) within the standard interpretation, or equivalently, that $E(x)$ is certainly true, hence true (certainly false, hence false) within the SR interpretation, by checking whether the state S of x belongs to the set $\mathcal{S}_E$ ($\mathcal{S}_E^{\perp}$). The empirical procedure required by this check is rather complex, since it does not reduce to a test of E on the physical object x, but consists in testing a huge number of physical objects in the state S by means of registrations belonging to E, in order to show that all of them yield outcome 1 (0) (it has been proven elsewhere[(26)] that this procedure actually tests a quantified statement, or a second order physical property).

We conclude by noticing that truth and empirical provability of truth coincide within the standard interpretation of QM, which expresses the verificationist position that characterizes this interpretation. On the contrary, within the SR interpretation of QM the concepts of truth and empirical provability of truth are different, in accordance with the well known distinction between truth and epistemic accessibility of truth in classical logic.

3 QL as a pragmatic language

We aim to show in this section that physical QL can be recovered as a pragmatic language in the sense established in Ref. 27. It is noteworthy that, by weakening slightly the assumptions introduced in Ref. 27, one could perform this task without choosing between the standard and the SR interpretation of QM (see footnotes 8 and 9). We adopt however the SR interpretation in this section, since we maintain that the verificationist attitude of the standard interpretation is epistemologically and philosophically doubtful, but we point out by means of footnotes the simple changes to be introduced in order to attain the same results within the standard interpretation.

3.1 The general pragmatic language $\mathcal{L}^P$

Let us summarize briefly the construction of the general pragmatic language $\mathcal{L}^P$ introduced in Ref. 27.

The alphabet $\mathcal{A}^P$ of $\mathcal{L}^P$ contains as *descriptive signs* the propositional letters p, q, r,...; as *logical-semantic signs* the connectives $\urcorner$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$; as *logical-*

pragmatic signs the assertion sign $\vdash$ and the connectives N, K, A, C and E; as *auxiliary signs* the round brackets (.). The set ψ_R of all *radical formulas* (*rfs*) of $\mathcal{L}^P$ is made up by all formulas constructed by means of descriptive and logical-semantic signs, following the standard recursive rules of classical propositional logic (a rf consisting of a propositional letter only is then called *atomic*). The set ψ_A of all *assertive formulas* (*afs*) of $\mathcal{L}^P$ is made up by all rfs preceded by the assertive sign $\vdash$ (*elementary* afs), plus all formulas constructed by using elementary afs and following standard recursive rules in which N, K, A, C and E take the place of $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$, respectively.

A *semantic interpretation* of $\mathcal{L}^P$ is then defined as a pair $(\{1,0\},\sigma)$, where σ is an *assignment function* which maps ψ_R onto the set $\{1,0\}$ of *truth values* (1 standing for *true* and 0 for *false*), following the standard truth rules of classical propositional calculus.

Whenever a semantic interpretation σ is given, a *pragmatic interpretation* of $\mathcal{L}^P$ is defined as a pair $(\{J,U\},\pi_\sigma)$, where π_σ is a *pragmatic evaluation function* which maps ψ_A onto the set $\{J,U\}$ of *justification values* following *justification rules* which refer to σ and are based on the informal properties of the metalinguistic concept of proof in natural languages. In particular, the following justification rules hold.

JR$_1$. *Let $\alpha \in \psi_R$; then, $\pi_\sigma(\vdash \alpha) = J$ iff a proof exists that α is true, i.e., that $\sigma(\alpha) = 1$ (hence, $\pi_\sigma(\vdash \alpha) = U$ iff no proof exists that α is true).*

JR$_2$. *Let $\delta \in \psi_A$; then, $\pi_\sigma(N\delta) = J$ iff a proof exists that δ is unjustified, i.e., that $\pi_\sigma(\delta) = U$.*

JR$_3$. *Let δ_1, $\delta_2 \in \psi_A$; then,*
(i) $\pi_\sigma(\delta_1 K \delta_2) = J$ iff $\pi_\sigma(\delta_1) = J$ and $\pi_\sigma(\delta_2) = J$,
(ii) $\pi_\sigma(\delta_1 A \delta_2) = J$ iff $\pi_\sigma(\delta_1) = J$ or $\pi_\sigma(\delta_2) = J$,
(iii) $\pi_\sigma(\delta_1 C \delta_2) = J$ iff a proof exists that $\pi_\sigma(\delta_2) = J$ whenever $\pi_\sigma(\delta_1) = J$,
(iv) $\pi_\sigma(\delta_1 E \delta_2) = J$ iff $\pi_\sigma(\delta_1 C \delta_2) = J$ and $\pi_\sigma(\delta_2 C \delta_1) = J$.

Furthermore, the following *correctness criterion* holds in $\mathcal{L}^P$.

CC. *Let $\alpha \in \psi_R$; then, $\pi_\sigma(\vdash \alpha) = J$ implies $\sigma(\alpha) = 1$.*

Finally, the set of all pragmatic evaluation functions that can be associated with a given semantic interpretation σ is denoted by Π_σ.

3.2 The quantum pragmatic language $\mathcal{L}_Q^P$

The quantum pragmatic language $\mathcal{L}_Q^P$ that we want to introduce here is obtained by specializing syntax, semantics and pragmatics of $\mathcal{L}^P$. Let us begin with the syntax. We introduce the following assumptions on $\mathcal{L}_Q^P$.

A$_1$. *The propositional letters p, q, ... are substituted by the symbols $E(x)$, $F(x)$, ..., with E, F, ... $\in \mathcal{E}$.*

A$_2$. *The set ψ_R^Q of all rfs of $\mathcal{L}_Q^P$ reduces to the set of all atomic rfs of $\mathcal{L}_Q^P$ (in different words, if α is a rf of $\mathcal{L}_Q^P$, then $\alpha = E(x)$, with $E \in \mathcal{E}$).*

A$_3$. *Only the logical-pragmatic signs* $\vdash$, N, K *and* A *appear in the afs of* $\mathcal{L}_Q^P$.

The substitution in A$_1$ aims to suggest the *intended interpretation* that we adopt in the following. To be precise, the rfs $E(x)$, $F(x)$, ... are interpreted as sentences stating that the physical object x has the properties E, F, ..., respectively (Sec. 2.4).

The restriction in A$_2$ aims to select rfs that are interpreted as *testable* sentences, i.e., sentences stating testable physical properties (Sec. 2.1), so that physical procedures exist for testing their truth values (which may not occur in the case of a rf of the form, say, $E(x) \vee F(x)$; note that a similar restriction has been introduced in Ref. 27 when recovering intuitionistic propositional logic within $\mathcal{L}^P$).

The restriction in A$_3$ is introduced for the sake of simplicity, since only the pragmatic connectives N, K and A are relevant for our goals in this paper.

Because of A$_1$, A$_2$ and A$_3$, the set ψ_A^Q of afs of $\mathcal{L}_Q^P$ is made up by all formulas constructed by means of the following recursive rules.

(i) *Let* $E(x)$ *be a rf. Then* $\vdash E(x)$ *is an af.*

(ii) *Let* δ *be an af. Then,* $N\delta$ *is an af.*

(iii) *Let* δ_1 *and* δ_2 *be afs. Then,* $\delta_1 K \delta_2$ *and* $\delta_1 A \delta_2$ *are afs.*

Let us come now to the semantics of $\mathcal{L}_Q^P$. We introduce the following assumption on $\mathcal{L}_Q^P$.

A$_4$. *Every assignment function* σ *defined on* ψ_R^Q *is induced by an interpretation* ξ *of the variable* x *that appears in the rfs into a universe* $\mathcal{U}$ *of physical objects, hence* $\sigma = \sigma(\xi)$ *and the values of* σ *on* ψ_R^Q *are consistent with (not necessarily determined by) the laws of QM within the intended interpretation established above.*

Let us comment briefly on assumption A$_4$.

Firstly, note that the interpretation ξ was understood in Sec. 2.1, when we introduced the informal expression "the physical object x is in the state S".

Secondly, observe that the requirement that $\sigma = \sigma(\xi)$ be consistent with the laws of QM (briefly, *QM-consistent*) obviously follows from the fact that these laws, via intended interpretation, establish relations among the truth values of elementary rfs of $\mathcal{L}_Q^P$ whenever a specific physical object is considered. We denote by Σ in the following the set of all QM-consistent assigment functions.

Thirdly, note that, since $\sigma = \sigma(\xi)$, there may be many interpretations of the variable x that lead to the same assigment function.

Finally, observe that the universe $\mathcal{U}$ can be partitioned into (disjoint) subsets of physical objects, each of which consists of physical objects in the same state (different subsets corresponding to different states). Thus, specifying the state S of x means requiring that the interpretation ξ of x that is considered maps x on a physical object in the subset corresponding to the state S, hence it singles out a subclass $\Sigma_S \subset \Sigma$ of assigment functions. All functions in Σ_S assign truth value 1 (0) to a sentence $E(x) \in \psi_R^Q$ whenever $E \in \mathcal{E}_S$ ($\mathcal{E}_S^\perp$), while the truth

values assigned by different functions in Σ_S to $E(x)$ may differ if $E \notin \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$.[8]

Let us come now to the pragmatics of $\mathcal{L}_Q^P$. We introduce the following assumption on $\mathcal{L}_Q^P$.

A$_5$. *Let a mapping ξ be given which interpretes the variable x in the rfs of $\mathcal{L}_Q^P$ on a physical object in the state S. A proof that the rf $E(x)$ is true (false) consists in performing one of the empirical procedures mentioned in Sec. 2.6 and showing that $E \in \mathcal{E}_S$ ($E \in \mathcal{E}_S^{\perp}$).*

Assumption A$_5$ is obviously suggested by the intended interpretation discussed above. Taking into account A$_1$ and JR$_1$ in Sec. 3.1, it implies the following statement.

P. *Let $E(x)$ be a rf of $\mathcal{L}_Q^P$, let ξ be an interpretation of the variable x on a physical object in the state S, and let S_E be defined as in Sec. 2.2. Then,*
$\pi_{\sigma(\xi)}(\vdash E(x)) = J$ *iff* $S \in S_E$,
$\pi_{\sigma(\xi)}(\vdash E(x)) = U$ *iff* $S \notin S_E$.

The above result specifies $\pi_{\sigma(\xi)}$ on the set of all elementary afs of $\mathcal{L}_Q^P$ and shows that it depends only on the state S. Hence, we write π_S in place of $\pi_{\sigma(\xi)}$ in the following (for the sake of brevity, we also agree to use the intuitive statement "the physical object x is in the state S" introduced in Sec. 2.1 in place of the more rigorous statement "the variable x is interpreted on a physical object in the state S").

Statement P provides the starting point for introducing a *set-theoretical pragmatics* for $\mathcal{L}_Q^P$, as follows.

Firstly, we introduce a mapping

$f : \delta \in \psi_A^Q \longrightarrow \mathcal{S}_\delta \in \mathcal{P}(\mathcal{S})$

which associates a *pragmatic extension* $\mathcal{S}_\delta$ with every assertive formula $\delta \in \psi_A^Q$, defined by the following recursive rules.

(i) *For every* $E(x) \in \psi_R^Q$, $f(\vdash E(x)) = S_{\vdash E(x)} = S_E$.
(ii) *For every* $\delta \in \psi_A^Q$, $f(N\delta) = S_{N\delta} = S_\delta^{\perp}$.
(iii) *For every* $\delta_1, \delta_2 \in \psi_A^Q$, $f(\delta_1 K\, \delta_2) = \mathcal{S}_{\delta_1 K \delta_2} = \mathcal{S}_{\delta_1} \cap \mathcal{S}_{\delta_2}$.
(iv) *For every* $\delta_1, \delta_2 \in \psi_A^Q$, $f(\delta_1 A\, \delta_2) = S_{\delta_1 A \delta_2} = S_{\delta_1} \cup S_{\delta_2}$.

Secondly, we rewrite statement P above substituting $\mathcal{S}_{\vdash E(x)}$ to $\mathcal{S}_E$ in it.

P$'$. *Let $\vdash E(x)$ be an elementary af of $\mathcal{L}_Q^P$ and let x be in the state S. Then,*
$\pi_S(\vdash E(x)) = J$ *iff* $S \in S_{\vdash E(x)}$,
$\pi_S(\vdash E(x)) = U$ *iff* $S \notin S_{\vdash E(x)}$.

Thirdly, we note that statement P$'$ defines the pragmatic evaluation function π_S on all elementary afs of $\mathcal{L}_Q^P$.

[8]Assumption A$_4$ can be stated unchanged whenever the standard interpretation of QM is adopted instead of the SR interpretation. In this case, however, for every ξ, $\sigma(\xi)$ is defined only on a subset of rfs, not on the whole ψ_R^Q (which requires a weakening of the assumptions on σ if one wants to recover this case within the general perspective in Sec. 3.1). Furthermore, Σ_S reduces to a singleton. Indeed, for every interpretation ξ, a state $S = S(\xi)$ exists such that $\xi(x) \in S$. Then, $\sigma(\xi)$ is defined on a rf $E(x)$ iff $E \in \mathcal{E}_S \cup \mathcal{E}_S^{\perp}$ (Sec. 2.4), and does not change if ξ is substituted by an interpretation ξ' such that $\xi'(x) \in S$.

Finally, for every $S \in \mathcal{S}$, we extend π_S from the set of all elementary afs of $\mathcal{L}_Q^P$ to the set ψ_A^Q of all afs of $\mathcal{L}_Q^P$ bearing in mind JR_2 and JR_3 in Sec. 3.1, hence introducing the following recursive rules.

(i) *For every* $\delta \in \psi_A^Q$, $\pi_S(N\delta) = J$ *iff* $S \in S_{N\delta} = S_\delta^\perp$.

(ii) *For every* $\delta_1, \delta_2 \in \psi_A^Q$, $\pi_S(\delta_1 K\ \delta_2) = J$ *iff* $S \in S_{\delta_1 K \delta_2} = S_{\delta_1} \cap S_{\delta_2}$.

(iii) *For every* $\delta_1, \delta_2 \in \psi_A^Q$, $\pi_S(\delta_1 A\ \delta_2) = J$ *iff* $S \in S_{\delta_1 A \delta_2} = S_{\delta_1} \cup S_{\delta_2}$.

The above procedure defines, for every $S \in \mathcal{S}$, a pragmatic evaluation function

$\pi_S : \delta \in \psi_A^Q \longrightarrow \pi_S(\delta) \in \{J, U\}$

which provides a set-theoretical pragmatics for $\mathcal{L}_Q^P$, as stated.

3.3 On the notion of justification in $\mathcal{L}_Q^P$

The notion of justification introduced in Sec. 3.2 is basic in our approach and must be clearly understood. So we devote this section to comments on it.

Whenever an elementary af $\vdash E(x)$ of $\mathcal{L}_Q^P$ is considered, the notion of justification obviously coincides with the notion of existence of an empirical proof of the truth of $E(x)$ because of assumption A_5 and proposition P in Sec. 3.2, which fits in with JR_1 in Sec. 3.1.

Whenever molecular afs of $\mathcal{L}^P$ are considered, one can grasp intuitively the meaning of the notion of justification for them by considering simple instances. Indeed, let $E(x)$ be a rf and let x be in the state S. We get

$\pi_S(N \vdash E(x)) = J$ iff $S \in \mathcal{S}_E^\perp$,

which means, shortly, that it is justified to assert that $E(x)$ cannot be asserted iff MQ entails that the truth value of $E(x)$ is *false* for every x in the state S. This result, of course, fits in with JR_2 in Sec. 3.1.

Furthermore, let $E(x)$ and $F(x)$ be rfs, and let x be in the state S. We get

$\pi_S(\vdash E(x) K \vdash F(x)) = J$ iff $S \in \mathcal{S}_E \cap \mathcal{S}_F$,

$\pi_S(\vdash E(x) A \vdash F(x)) = J$ iff $S \in \mathcal{S}_E \cup \mathcal{S}_F$.

The first equality shows that asserting $E(x)$ and $F(x)$ conjointly is justified iff both assertions are justified. The second equality shows that asserting $E(x)$ or asserting $F(x)$ is justified iff one of these assertions is justified. Both these results, of course, fit in with JR_3 in Sec. 3.1.

We add that

$\pi_S(\vdash E(x)) = J$ implies $\pi_S(N \vdash E(x)) = U$

and

$\pi_S(N \vdash E(x)) = J$ implies $\pi_S(\vdash E(x)) = U$

since $\mathcal{S}_E \cap \mathcal{S}_E^\perp = \emptyset$. Nevertheless,

$\pi_S(\vdash E(x)) = U$ and $\pi_S(N \vdash E(x)) = U$ iff $S \notin \mathcal{S}_E \cup \mathcal{S}_E^\perp$,

which shows that a *tertium non datur* principle does not hold for the pragmatic connective N in $\mathcal{L}_Q^P$ (it has already been proved in Ref. 27 that this principle does not hold in the general language $\mathcal{L}^P$).

It is also interesting to note that the justification values of different elementary afs, say $\vdash E(x)$ and $\vdash F(x)$, must be different for some state S, since $\mathcal{S}_E \neq \mathcal{S}_F$ if $E \neq F$ (Sec. 2.2), hence $\mathcal{S}_{\vdash E(x)} \neq \mathcal{S}_{\vdash F(x)}$.

Finally, we remind that the general theory of $\mathcal{L}^P$ associates an assignment function σ with a set Π_σ of pragmatic evaluation functions (Sec. 3.1), hence this also occurs within $\mathcal{L}_Q^P$. One may then wonder whether Π_σ is necessarily nonvoid and, if this is the case, whether it may contain more than one pragmatic evaluation function. In order to answer these questions, let us consider an interpretation ξ of the variable x that maps x on a physical object in the state S. Then, ξ determines a unique assignment function $\sigma(\xi)$ and a unique pragmatic evaluation function associated with it, that we have denoted by π_S, for it depends only on the state S. Since every assigment function in Σ is induced by an interpretation ξ because of A_4 in Sec. 3.2, this proves that Π_σ is necessarily nonvoid for every $\sigma \in \Sigma$. Moreover, note that an interpretation ξ' of x may exist within the SR interpretation of QM that maps x on a physical object in the state S', with $S' \neq S$, yet such that $\sigma(\xi') = \sigma(\xi)$. The pragmatic evaluation functions π_S and $\pi_{S'}$ are then different, but they are both associated with the assignment function $\sigma = \sigma(\xi) = \sigma(\xi')$, so that they both belong to Π_σ. Hence, Π_σ may contain many pragmatic evaluation functions.[9]

3.4 Pragmatic validity and order in $\mathcal{L}_Q^P$

Coming back to the general language $\mathcal{L}^P$, we remind that a notion of pragmatic validity (invalidity) is introduced in it by means of the following definition.

Let $\delta \in \psi_A$. Then, δ is pragmatically valid*, or* p-valid *(*pragmatically invalid*, or* p-invalid*) iff for every $\sigma \in \Sigma$ and $\pi_\sigma \in \Pi_\sigma$, $\pi_\sigma(\delta) = J$ ($\pi_\sigma(\delta) = U$).*

By using the notions of justification in $\mathcal{L}_Q^P$, one can translate the notion of p-validity (p-invalidity) within $\mathcal{L}_Q^P$ as follows.

Let $\delta \in \psi_A^Q$. Then, δ is p-valid (p-invalid) iff, for every $S \in \mathcal{S}$, $\pi_S(\delta) = J$ ($\pi_S(\delta) = U$).

The notion of p-validity (p-invalidity) can then be characterized as follows.

Let $\delta \in \psi_A^Q$. Then, δ is p-valid (p-invalid) iff $\mathcal{S}_\delta = \mathcal{S}$ ($\mathcal{S}_\delta = \emptyset$).

The set of all p-valid afs plays in $\mathcal{L}_Q^P$ a role similar to the role of tautologies in classical logic, and some afs in it can be selected as axioms if one tries to construct a p-correct and p-complete calculus for $\mathcal{L}_Q^P$. We will not deal, however, with this topic in the present paper.

Furthermore, let us observe that a binary relation can be introduced in the general language $\mathcal{L}^P$ by means of the following definition.

[9] Assumption A_5 in Sec. 3.2 can be stated unchanged if the standard interpretation of QM is adopted instead of the SR interpretation. In this case, however, it is impossible that a mapping ξ' exists such that $\xi'(x) \in S'$, with $S \neq S'$ and $\sigma(\xi) = \sigma(\xi')$, since $\sigma(\xi)$ and $\sigma(\xi')$ are defined on different domains ($\mathcal{E}_S \cup \mathcal{E}_S^\perp$ and $\mathcal{E}_{S'} \cup \mathcal{E}_{S'}^\perp$, respectively). Hence, an assigment function σ is associated with a unique state S, and Π_σ reduces to the singleton $\{\pi_S\}$.

For every δ_1, $\delta_2 \in \psi_A$, $\delta_1 \prec \delta_2$ iff a proof exists that δ_2 is justified whenever δ_1 is justified (equivalently, $\delta_1 \prec \delta_2$ iff $\delta_1 C \delta_2$ is justified).

The set-theoretical pragmatics introduced in Sec. 3.2 allows one to translate the above definition in $\mathcal{L}_Q^P$ as follows.

For every δ_1, $\delta_2 \in \psi_A^Q$, $\delta_1 \prec \delta_2$ iff for every $S \in \mathcal{S}$, $\pi_S(\delta_1) = J$ implies $\pi_S(\delta_2) = J$.

The binary relation $\prec$ can then be characterized as follows.

For every δ_1, $\delta_2 \in \psi_A^Q$, $\delta_1 \prec \delta_2$ iff $\mathcal{S}_{\delta_1} \subset \mathcal{S}_{\delta_2}$.

The relation $\prec$ is obviously a pre-order relation on ψ_A^Q, hence it induces canonically an equivalence relation $\approx$ on ψ_A^Q, defined as follows.

For every δ_1, $\delta_2 \in \psi_A^Q$, $\delta_1 \approx \delta_2$ iff $\delta_1 \prec \delta_2$ and $\delta_2 \prec \delta_1$.

The equivalence relation $\approx$ can then be characterized as follows.

For every δ_1, $\delta_2 \in \psi_A^Q$, $\delta_1 \approx \delta_2$ iff $\mathcal{S}_{\delta_1} = \mathcal{S}_{\delta_2}$.

3.5 Decidability versus justifiability in $\mathcal{L}_Q^P$

We have commented rather extensively in Sec. 3.3 on the notion of justification formalized in $\mathcal{L}_Q^P$, for every $S \in \mathcal{S}$, by the pragmatic evaluation function π_S. It must still be noted, however, that the definition of π_S on all afs in ψ_A^Q does not grant that an empirical procedure of proof exists which allows one to establish, for every $S \in \mathcal{S}$, the justification value of every af of $\mathcal{L}_Q^P$. In order to understand how this may occur, note that the notion of empirical proof is defined by A_5 for atomic rfs of $\mathcal{L}_Q^P$ and makes explicit reference, for every $E(x) \in \psi_R^Q$, to the closed subset $\mathcal{S}_E \in \mathcal{L}(\mathcal{S})$ associated with E by the function ρ introduced in Sec. 2.2. Basing on this notion, the justification value $\pi_S(\vdash E(x))$ of an elementary af $\vdash E(x) \in \psi_A^Q$ can be determined by means of the same empirical procedure, making reference to the closed subset $\mathcal{S}_{\vdash E(x)} = \mathcal{S}_E$ associated to $\vdash E(x)$ by the function f (Sec. 3.2). Yet, whenever π_S is recursively defined on the whole ψ_A^Q, new subsets of states are introduced (as $\mathcal{S}_{\delta_1} \cup \mathcal{S}_{\delta_2}$) which do not necessarily belong to $\mathcal{L}(\mathcal{S})$. If an af δ is associated by f with a subset that does not belong to $\mathcal{L}(\mathcal{S})$, no empirical procedure exists in QM which allows one to determine the justification value $\pi_S(\delta)$.

We are thus led to introduce the subset $\psi_{AD}^Q \subset \psi_A^Q$ of all *pragmatically decidable*, or *p-decidable*, afs of $\mathcal{L}_Q^P$. An af δ of $\mathcal{L}_Q^P$ is p-decidable iff an empirical procedure of proof exists which allows one to establish whether δ is justified or unjustified, whatever the state S of x may be.

Because of the remark above, the subset of all p-decidable afs of $\mathcal{L}_Q^P$ can be characterized as follows.

$\psi_{AD}^Q = \{\delta \in \psi_A^Q \mid \mathcal{S}_\delta \in \mathcal{L}(\mathcal{S})\}$.

Let us discuss some criteria for establishing whether a given af $\delta \in \psi_A^Q$ belongs to ψ_{AD}^Q.

C_1. *All elementary afs of ψ_A^Q belong to ψ_{AD}^Q.*

C_2. *If* $\delta \in \psi^Q_{AD}$, *then* $N\delta \in \psi^Q_{AD}$

Indeed, $S_\delta \in \mathcal{L}(\mathcal{S})$ implies $S^\perp_\delta \in \mathcal{L}(\mathcal{S})$.

C_3. *If* δ_1, $\delta_2 \in \psi^Q_{AD}$, *then* $\delta_1 K\ \delta_2 \in \psi^Q_{AD}$

Indeed, $S_{\delta_1} \in \mathcal{L}(\mathcal{S})$ and $S_{\delta_2} \in \mathcal{L}(\mathcal{S})$ imply $S_{\delta_1} \cap S_{\delta_2} \in \mathcal{L}(\mathcal{S})$, since $S_{\delta_1} \cap S_{\delta_2} = S_{\delta_1} \Cap S_{\delta_2}$ because of known properties of the lattice $(\mathcal{L}(\mathcal{S}), \subset)$ (Sec. 2.2).

C_4. *If* δ_1, $\delta_2 \in \psi^Q_{AD}$, *then* $\delta_1 A\ \delta_2$ *may belong or not to* ψ^Q_{AD}. *To be precise, it belongs to* ψ^Q_{AD} *iff* $S_{\delta_1} \subset S_{\delta_2}$ *or* $S_{\delta_2} \subset S_{\delta_1}$

Indeed, $S_{\delta_1} \cup S_{\delta_2} \in \mathcal{L}(\mathcal{S})$ or, equivalently, $S_{\delta_1} \cup S_{\delta_2} = S_{\delta_1} \Cup S_{\delta_2}$, iff one of the conditions in C_4 is satisfied.

It is apparent from criteria C_2 and C_3 that ψ^Q_{AD} is closed with respect to the pragmatic connectives N and K, in the sense that $\delta \in \psi^Q_{AD}$ implies $N\delta \in \psi^Q_{AD}$, and δ_1, $\delta_2 \in \psi^Q_{AD}$ implies $\delta_1 K \delta_2 \in \psi^Q_{AD}$. On the contrary, ψ^Q_{AD} is not closed with respect to A, since it may occur that $\delta_1 A\ \delta_2 \notin \psi^Q_{AD}$ even if δ_1, $\delta_2 \in \psi^Q_{AD}$. In order to obtain a closed subset of afs of $\mathcal{L}^P_Q$, one can consider the set

$$\phi^Q_{AD} = \{\delta \in \psi^Q_A \mid \text{the pragmatic connective } A \text{ does not occur in } \delta\}.$$

The set ϕ^Q_{AD} obviously contains all elementary afs of $\mathcal{L}^P_Q$, plus all afs of ψ^Q_A in which only the pragmatic connectives N and K occur. We can thus consider a sublanguage of $\mathcal{L}^P_Q$ whose set of afs reduces to ϕ^Q_{AD}. This new language is relevant since all its afs are p-decidable, hence we call it *the p-decidable sublanguage* of $\mathcal{L}^P_Q$ and denote it by $\mathcal{L}^P_{QD}$.

3.6 The p-decidable sublanguage $\mathcal{L}^P_{QD}$

As we have anticipated in the Introduction, we aim to show in this paper that the sublanguage $\mathcal{L}^P_{QD}$ has the structure of a physical QL, hence it provides a new pragmatic interpretation of this relevant physical structure. However, this interpretation will be more satisfactory from an intuitive viewpoint if we endow $\mathcal{L}^P_{QD}$ with some further derived pragmatic connectives which can be made to correspond with connectives of physical QL. To this end, we introduce the following definitions.

D_1. *We call* quantum pragmatic disjunction *the connective* A_Q *defined as follows.*

For every δ_1, $\delta_2 \in \phi^Q_{AD}$, $\delta_1 A_Q \delta_2 = N((N\delta_1)K(N\delta_2))$.

D_2. *We call* quantum pragmatic implication *the connective* I_Q *defined as follows.*

For every δ_1, $\delta_2 \in \phi^Q_{AD}$, $\delta_1 I_Q \delta_2 = (N\delta_1)A_Q(\delta_1 K \delta_2)$.

Let us discuss the justification rules which hold for afs in which the new connectives A_Q and I_Q occur.

By using the function f introduced in Sec. 3.2 we get (since the set-theoretical operation $\cap$ coincides with the lattice operation $\Cap$ in $(\mathcal{L}(\mathcal{S}), \subset)$, see Sec. 2.2),

$$\mathcal{S}_{\delta_1 A_Q \delta_2} = \mathcal{S}^\perp_{(N\delta_1)K(N\delta_2)} = (\mathcal{S}_{N\delta_1} \cap \mathcal{S}_{N\delta_2})^\perp = (\mathcal{S}^\perp_{\delta_1} \Cap \mathcal{S}^\perp_{\delta_2})^\perp = (\mathcal{S}_{\delta_1} \Cup \mathcal{S}_{\delta_2}).$$

Hence, for every $S \in \mathcal{S}$,

$\pi_S(\delta_1 A_Q \delta_2) = J$ iff $S \in \mathcal{S}_{\delta_1} \Cup \mathcal{S}_{\delta_2}$.

Let us come to the quantum pragmatic implication I_Q. By using the definition of A_Q, one gets

$\delta_1 I_Q \delta_2 = N((NN\delta_1)K(N(\delta_1 K \delta_2))$.

By using the function f and the above result about A_Q, one then gets

$\mathcal{S}_{\delta_1 I_Q \delta_2} = \mathcal{S}_{N\delta_1} \Cup \mathcal{S}_{\delta_1 K \delta_2} = \mathcal{S}_{\delta_1}^{\perp} \Cup (\mathcal{S}_{\delta_1} \Cap \mathcal{S}_{\delta_2})$.

It follows that, for every $S \in \mathcal{S}$,

$\pi_S(\delta_1 I_Q \delta_2) = J$ iff $S \in \mathcal{S}_{\delta_1}^{\perp} \Cup (\mathcal{S}_{\delta_1} \Cap \mathcal{S}_{\delta_2})$.

Let us observe now that $\mathcal{L}_{QD}^P$ obviously inherits the notions of p-validity and order defined in $\mathcal{L}_Q^P$ (Sec. 3.4). Hence, we can illustrate the role of the connective I_Q within $\mathcal{L}_{QD}^P$ by means of the following *pragmatic deduction lemma*.

PDL. *Let δ_1, $\delta_2 \in \phi_{AD}^Q$. Then, $\delta_1 \prec \delta_2$ iff for every $S \in S$, $\pi_S(\delta_1 I_Q \delta_2) = J$ (equivalently, iff $\delta_1 I_Q \delta_2$ is p-valid).*

Proof. The following sequence of equivalences holds.

For every $S \in \mathcal{S}$, $\pi_S(\delta_1 I_Q \delta_2) = J$ iff for every $S \in \mathcal{S}$, $S \in \mathcal{S}_{\delta_1}^{\perp} \Cup (\mathcal{S}_{\delta_1} \Cap \mathcal{S}_{\delta_2})$ iff $\mathcal{S}_{\delta_1}^{\perp} \Cup (\mathcal{S}_{\delta_1} \Cap \mathcal{S}_{\delta_2}) = \mathcal{S}$ iff $\mathcal{S}_{\delta_1} \Cap \mathcal{S}_{\delta_2} = \mathcal{S}_{\delta_1}$ iff $\mathcal{S}_{\delta_1} \subset \mathcal{S}_{\delta_2}$ iff $\delta_1 \prec \delta_2$.■

PDL shows that the quantum pragmatic implication I_Q plays within $\mathcal{L}_{QD}^P$ a role similar to the role of material implication in classical logic.

3.7 Interpreting QL onto $\mathcal{L}_{QD}^P$

In order to show that the physical QL $(\mathcal{E}, \prec)$ introduced in Sec. 2.2 can be interpreted into $\mathcal{L}_{QD}^P$, a further preliminary step is needed. To be precise, let us make reference to the preorder introduced on ψ_A^Q in Sec. 3.4 and consider the pre-ordered set $(\phi_{AD}^Q, \prec)$ of all afs of $\mathcal{L}_{QD}^P$. Furthermore, let us denote by $\approx$ (by abuse of language) the restriction of the equivalence relation introduced on ψ_A^Q in Sec. 3.4 to ϕ_{AD}^Q, and let us denote by $\prec$ (again by abuse of language) the partial order induced on $\phi_{AD}^Q/\approx$ by the preorder defined on ϕ_{AD}^Q. Then, let us show that $(\phi_{AD}^Q/\approx, \prec)$ is order isomorphic to $(\mathcal{L}(\mathcal{S}), \subset)$.

Let us consider the mapping

$f_{\approx} : [\delta]_{\approx} \in \psi_{AD}^Q/\approx \longrightarrow \mathcal{S}_\delta \in \mathcal{L}(\mathcal{S})$.

This mapping is obviously well defined because of the characterization of $\approx$ in Sec. 3.4. Furthermore, the following statements hold.

(i) *For every $\delta \in \phi_{AD}^Q$, one and only one elementary af $\vdash E(x)$ exists such that $\vdash E(x) \in [\delta]_{\approx}$.*

(ii) *The mapping $f_{\approx}$ is bijective.*

(iii) *For every δ_1, $\delta_2 \in \phi_{AD}^Q$, $[\delta_1]_{\approx} \prec [\delta_2]_{\approx}$ iff $S_{\delta_1} \subset S_{\delta_2}$.*

Let us prove (i). Consider $[\delta]_{\approx}$. Since $\mathcal{S}_\delta \in \mathcal{L}(\mathcal{S})$ and ρ is bijective (Sec. 2.2), a property $E \in \mathcal{E}$ exists such that $E = \rho^{-1}(\mathcal{S}_\delta)$, hence $\mathcal{S}_\delta = \mathcal{S}_E$. It follows that

$[\delta]_{\approx}$ contains the af $\vdash E(x)$, for $\mathcal{S}_{\vdash E(x)} = \mathcal{S}_E$ (Sec. 3.2). Moreover, $[\delta]_{\approx}$ does not contain any further elementary af. Indeed, let $\vdash F(x)$ be an elementary af of ϕ^Q_{AD} with $E \neq F$: then, $\mathcal{S}_E \neq \mathcal{S}_F$, hence $\mathcal{S}_{\vdash E(x)} \neq \mathcal{S}_{\vdash F(x)}$, which implies $\vdash F(x) \notin [\delta]_{\approx}$. Thus, statement (i) is proved.

The proofs of statements (ii) and (iii) are then immediate. Indeed, statement (ii) follows from (i) and from the definition of $f_{\approx}$, while statement (iii) follows from (ii) and from the definition of $\prec$ on $\phi^Q_{AD}/\approx$.

Because of (ii) and (iii), the poset $(\phi^Q_{AD}/\approx, \prec)$ is order-isomorphic to $(\mathcal{L}(\mathcal{S}), \subset)$, as stated.

Let us come now to physical QL. We have seen in Sec. 2.2 that $(\mathcal{L}(\mathcal{S}), \subset)$ is order-isomorphic to $(\mathcal{E}, \prec)$. We can then conclude that $(\mathcal{E}, \prec)$ is order-isomorphic to $(\phi^Q_{AD}/\approx, \prec)$, which provides the desired interpretation of a physical QL into $\mathcal{L}^P_{QD}$.

Let us comment briefly on the pragmatic interpretation of physical QL provided above.

Firstly, we note that our interpretation maps $\mathcal{E}$ on the quotient set $\phi^Q_{AD}/\approx$, not onto ϕ^Q_{AD}. Yet, the set of the (well formed) formulas of the lattice $(\mathcal{E}, ^{\perp}, \Cap, \Cup)$ can be mapped bijectively onto ϕ^Q_{AD} by means of the mapping induced by the following formal correspondence.

(i) $E \in \mathcal{E} \longleftrightarrow \vdash E(x) \in \phi^Q_{AD}$.
(ii) $^{\perp} \longleftrightarrow N$
(iii) $\Cap \longleftrightarrow K$
(iv) $\Cup \longleftrightarrow A_Q$.

Thus, the formal language of QL, for which the lattice $(\mathcal{L}(\mathcal{S}), \subset)$ can be considered as an *algebraic semantics*,[(3)] can be substituted by the language $\mathcal{L}^P_{QD}$, for which $(\mathcal{L}(\mathcal{S}), \subset)$ can be considered as an *algebraic pragmatics* (by the way, we also note that the above correspondence makes I_Q correspond to a *Sasaki hook*, the role of which is well known in QL). This reinterpretation is relevant from a philosophical viewpoint, since it avoids all problems following from the standard concept of quantum truth (Sec. 2.4) considering physical QL as formalizing properties of a quantum concept of justification rather than a quantum concept of truth. This makes physical QL consistent also with the classical concept of truth adopted with the SR interpretation of QM (Sec. 2.5). Furthermore, as we have already observed in the Introduction, it places physical QL within a general *integrated perspective*, according to which non-Tarskian theories of truth can be integrated with Tarski's theory by reinterpreting them as theories of metalinguistic concepts that are different from truth (in the case of physical QL, the concept of *empirical justification* in QM).

Secondly, we observe that our interpretation has some consequences that are intuitively satisfactory. For instance, for every state $S \in \mathcal{S}$, it attributes a justification value to every af in ϕ^Q_{AD}, while it is well known that there are formulas in physical QL which have no truth value according to the standard interpretation of QL (Sec. 2.4).

3.8 Some remarks on a possible calculus for $\mathcal{L}_{QD}^{P}$

One may obviously wonder whether a calculus can be given for the language $\mathcal{L}_{QD}^{P}$ which is *pragmatically correct* (*p-correct*) and *pragmatically complete* (*p-complete*). This is not a difficult task if we limit ourselves to the general lattice structure of $(\phi_{AD}^{Q}/\approx, \prec)$. Indeed, a set of axioms and/or inference rules which endow $\phi_{AD}^{Q}/\approx$ of the structure of orthomodular lattice can be easily obtained by using the formal correspondence introduced in Sec. 3.7, since this correspondence allows one to translate the axioms and/or inference rules that are usually stated in order to provide a calculus for orthomodular QL into ϕ_{AD}^{Q} (of course, all the afs produced by this translation are p-valid afs of $\mathcal{L}_{QD}^{P}$). Here is a sample set of axioms of this kind (where, of course, δ, δ_1, δ_2 and δ_3 are afs of ϕ_{AD}^{Q}) obtained by translating a set of rules provided by Dalla Chiara and Giuntini.[32]

A$_1$. $\delta I_Q \delta$.

A$_2$. $(\ \delta_1 K\ \delta_2) I_Q \delta_1$.

A$_3$. $(\ \delta_1 K\ \delta_2) I_Q \delta_2$.

A$_4$. $\delta I_Q (NN\delta)$.

A$_5$. $(NN\delta) I_Q \delta$.

A$_6$. $((\delta_1 I_Q \delta_2) K (\delta_1 I_Q \delta_3)) I_Q (\delta_1 I_Q (\delta_2 K \delta_3))$.

A$_7$. $((\delta_1 I_Q \delta_2) K (\delta_2 I_Q \delta_3)) I_Q (\delta_1 I_Q \delta_3)$.

A$_8$. $(\delta_1 I_Q \delta_2) I_Q ((N\delta_2) I_Q (N\delta_1))$.

A$_9$. $(\delta_1 I_Q \delta_2) I_Q (\delta_2 I_Q (\delta_1 A_Q ((N\delta_1) K \delta_2)))$.

However, in order to obtain physical QL one needs a number of further axioms, since the structure of $(\mathcal{L}(\mathcal{H}), \subset)$ must be recovered (Sec. 2.2). Providing a complete calculus for such a structure is a much more complicate task, which must take into account a number of mathematical results in lattice theory (in particular, Soler's theorem[33]). Therefore we will not discuss this problem in the present paper.

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References

[1] M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974).

[2] M. Rédei, *Quantum Logic in Algebraic Approach* (Kluwer, Dordrecht, 1998).

[3] M. Dalla Chiara, R. Giuntini and R. Greechie, *Reasoning in Quantum Theory* (Kluwer, Dordrecht, 2004).

[4] S. Haak, *Philosophy of Logics*, (Cambridge University Press, Cambridge, 1978).

[5] A. Tarski, "The concept of truth in formalized languages," in Tarski, A., *Logic, Semantics, Metamathematics*, (Blackwell, Oxford, 1956).

[6] A. Tarski, "The semantic conception of truth and the foundations of semantics," in L. Linski (ed.), *Semantics and the Philosophy of Language* (University of Illinois Press, Urbana, 1944).

[7] A. Messiah, *Quantum Mechanics* (North Holland, Amsterdam, 1961).

[8] Cohen-Tannoudij, B. Dieu and F. Laloe, *Mecanique Quantique* (Hermann, Paris, 1973).

[9] W. Greiner, *Quantum Mechanics, an Introduction* (Verlag, Berlin, 1989).

[10] B. D'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Reading, MA, 1976).

[11] G. Ludwig, *Foundations of Quantum Mechanics I* (Springer, New York, 1983).

[12] C. Garola and S. Sozzo, "A semantic approach to the completeness problem in quantum mechanics," *Found. Phys.* **34**, 1249-1266 (2004).

[13] C. Piron, *Foundations of Quantum Physics* (Benjamin, Reading, MA, 1976).

[14] D. Aerts, "Description of many separated entities without the paradoxes encountered in quantum mechanics," *Found. Phys.* **12**, 1131-1170 (1982).

[15] J. S. Bell, "On the problem of hidden variables in quantum mechanics," *Rev. Mod. Phys.* **38**, 447-452 (1966).

[16] S. Kochen and E. P. Specker, "The problem of hidden variables in quantum mechanics," *J. Math. Mech.* **17**, 59-87 (1967).

[17] J. S. Bell, "On the Einstein-Podolski-Rosen paradox," *Physics* **1**, 195-200 (1964).

[18] N. D. Mermin, “Hidden variables and the two theorems of John Bell,” *Rev. Mod. Phys.* **65**, 803-815 (1993).

[19] C. Garola, “Classical foundations of quantum logic,” *Int. J. Theor. Phys.* **30**, 1-52 (1991).

[20] C. Garola, “Truth versus testability in quantum logic,” *Erkenntnis* **37**, 197-222 (1995).

[21] C. Garola and L. Solombrino, “The theoretical apparatus of semantic realism: A new language for classical and quantum physics,” *Found. Phys.* **26**, 1121-1164 (1996a).

[22] C. Garola and L. Solombrino, “Semantic realism versus EPR-like paradoxes: The Furry, Bohm-Aharonov and Bell paradoxes” *Found. Phys.* **26**, 1329-1356 (1996b).

[23] C. Garola, “Against ‘paradoxes’: A new quantum philosophy for quantum physics,” in *Quantum Physics and the Nature of Reality*, D. Aerts and J. Pykacz, eds. (Kluwer, Dordrecht,1999).

[24] C. Garola, “Objectivity versus nonobjectivity in quantum mechanics,” *Found. Phys.* **30**, 1539-1565 (2000).

[25] C. Garola, “A simple model for an objective interpretation of quantum mechanics,” *Found. Phys.* **32**, 1597-1615 (2002).

[26] C. Garola and J. Pykacz, “Locality and measurements within the SR model for an objective interpretation of quantum mechanics,” *Found. Phys.* **34**, 449-475 (2004).

[27] C. Dalla Pozza and C. Garola, “A pragmatic interpretation of intuitionistic propositional logic,” *Erkenntnis* **43**, 81-109 (1995).

[28] E. Beltrametti and G. Cassinelli, *The Logic of Quantum Mechanics* (Addison-Wesley, Reading, MA, 1981).

[29] G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics* (Benjamin, New York, 1963).

[30] J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley, Reading, MA, 1968).

[31] G. Birkhoff and J. von Neumann, “The logic of quantum mechanics,” *Ann. Math.* **37**, 823-843 (1936).

[32] M. Dalla Chiara and R. Giuntini, “La Logica Quantistica,” in G. Boniolo (ed.), *Filosofia della Fisica* (B. Mondadori, Milan, 1997)

[33] M. P. Soler, “Characterization of Hilbert spaces by orthomodular spaces,” *Comm. Algebra* **23**, 219-243 (1995).